ON THE PONTRYAGIN–STEENROD–WU THEOREM

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ABSTRACT

We present a short and direct proof (based on the Pontryagin–Thom construction) of the following Pontryagin–Steenrod–Wu theorem: (a) Let M be a connected orientable closed smooth $(n+1)$-manifold, $n \geq 3$. Define the degree map $\deg: \pi^n(M) \to H^n(M;\mathbb{Z})$ by the formula $\deg f = f^*[S^n]$, where $[S^n] \in H^n(M;\mathbb{Z})$ is the fundamental class. The degree map is bijective, if there exists $\beta \in H_2(M, \mathbb{Z}/2\mathbb{Z})$ such that $\beta \cdot w_2(M) \neq 0$. If such β does not exist, then deg is a 2-1 map; and (b) Let M be an orientable closed smooth $(n+2)$-manifold, $n \geq 3$. An element α lies in the image of the degree map if and only if $\rho_2\alpha \cdot w_2(M) = 0$, where $\rho_2\colon \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is reduction modulo 2.

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1. Introduction

Throughout this paper let M be a connected orientable closed smooth manifold of dimension $m = n+k$. Denote by $L_k(M)$ the set of k-dimensional framed links in M up to framed cobordism. By the Pontryagin–Thom construction, the set $L_k(M)$ is in 1–1 correspondence with the set $\pi^n(M) = [M; S^n]$ of continuous maps $M \to S^n$ up to homotopy. The main purpose of this paper is to describe $L_1(M) = \pi^n(M)$ for $k = 1$ and in the 'stable range' $n \geq 3$. The description of $\pi^n(M)$ was reduced in [Pon39] [Ste47] (see also [FoFu89; §30.3]) to a calculation with Steenrod squares, which was done by Wu (cf. [FoFu89; §30.2.D]).

In this paper we present a short proof of this Pontryagin–Steenrod–Wu classification theorem. There are reasons to believe that this is Pontryagin's original proof, which he never published, because he went straight ahead to the general case — when M is an arbitrary polyhedron (cf. Theorem 1.2 below and the remark after its formulation).

This classification is based on the notions of natural orientation on a framed link and degree of a framed link, defined as follows. Take a point x on a framed link L and let $f_1, \dots, f_n$ be the frame at this point. The basis $e_1, \dots, e_k$ of $T_x(L)$ is said to be **positive**, if the basis $e_1, \dots, e_k, f_1, \dots, f_n$ of $T_x(M)$ is positive. The **degree** $\deg L$ of L is the homology class (with integral coefficients) of positively oriented L. So we have a map

$$\deg \colon L_k(M) \to H_k(M; \mathbb{Z}).$$

The Hopf–Whitney theorem (1932–35) asserts that this map is bijective for $k = 0$ and surjective for $k = 1$.

THEOREM 1.1: (a) *Let M be a connected orientable closed smooth $(n+1)$-manifold, $n \geq 3$. The degree map $\deg \colon L_1(M) \to H_1(M; \mathbb{Z})$ is bijective, if there exists $\beta \in H_2(M, \mathbb{Z}/2\mathbb{Z})$ such that $\beta \cdot w_2(M) \neq 0$. If such β does not exist, then* deg *is a 2-1 map (i.e., each $\alpha \in H_1(M; \mathbb{Z})$ has exactly two preimages).*

(b) *Let M be an orientable closed smooth $(n+2)$-manifold, $n \geq 3$. Then an element α lies in the image of $\deg \colon L_2(M) \to H_2(M; \mathbb{Z})$ if and only if $\rho_2\alpha \cdot w_2(M) = 0$.*

Here $\cdot$ is the multiplication $H_k(M; \mathbb{Z}/2\mathbb{Z}) \times H^k(M; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ and $\rho_2 \colon \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is reduction modulo 2. However, in the proof of Theorem 1.1 it is convenient to replace the cohomological Stiefel–Whitney classes by their homological duals. These classes are denoted by the same letters w_i and $\bar{w}_i$, and their geometric definition (equivalent to other definitions) is recalled below.

Then $\cdot$ in the above (and in all the subsequent) formulae is to be understood as the intersection product $H_i(M) \times H_j(M) \to H_{i+j-m}(M)$.

THEOREM 1.2 (Pontryagin): (a) *Let M be a connected orientable closed smooth 3-manifold. Then for each $\alpha \in H_1(M;\mathbb{Z})$, $\deg^{-1}\alpha$ is in a one-to-one correspondence with $\mathbb{Z}/2\alpha \cap H_2(M;\mathbb{Z})$.*

(b) *Let M be an orientable closed smooth 4-manifold. Then an element α lies in the image of* $\deg\colon L_2(M) \to H_2(M;\mathbb{Z})$ *if and only if $\alpha \cdot \alpha = 0$.*

Theorem 1.2(b) can be proved analogously to our proof of Theorem 1.1(b) below. Our methods can perhaps be used to prove Theorem 1.2(a) which was stated without proof in [Pon39]. In fact, Theorem 1.2(a) was not included in [Pon39] (published in English), but only in the abstract (published in Russian), without any indication of its proof. This makes it even more important to have a published proof of this result.

2. Geometric definition of homology Stiefel–Whitney classes

Take a general position system of s tangent vector fields on M. Let $\Sigma \subset M$ be the set of points at which these vector fields are not linearly independent.

By transversality [DNF79; §10.3], Σ is a submanifold of M. The Stiefel–Whitney class $w_{m+1-s}(L) \in H_{s-1}(M;\mathbb{Z}/2\mathbb{Z})$ is the class of the submanifold Σ (this is the first obstruction to existence of a linear independent system of s tangent vector fields on M).

This definition can be easily generalized to the case when tangent vector fields in TM are replaced by vector fields in an arbitrary vector bundle with the base M. If $L \subset M$ is a submanifold, then such classes for the normal bundle of L in M and for the restriction of TM to L are denoted by $\bar{w}_2(L)$ and $w_2(M)|_L$, respectively.

We will also use relative versions of these classes. For example, suppose that $L \subset M$ is an l-submanifold with boundary and a system f of $m-l-1$ linearly independent normal vector fields is given on ∂L. Then we can extend f to an arbitrary general position system of normal vector fields on L.

Define $\bar{w}_2(L,f) \in H_{l-2}(L;\mathbb{Z}/2\mathbb{Z})$ to be the class of the $(l-2)$-submanifold, on which these extended vector fields are not linearly independent (this is the first obstruction to extension of f to a linear independent system on L). We will omit f from the notation, if no confusion could arise.

3. Proof of Theorem 1.1(b)

Take any $\alpha \in H_2(M;\mathbb{Z})$. The class α can be realized by an orientable 2-submanifold $L \subset M$. Clearly, $\alpha \in \operatorname{Im}\deg$ if and only if some such L can be framed.

We can consider only connected L. Indeed, if some disconnected L can be framed, then the submanifold, which is the connected sum of all connected components of L, can also be framed and realizes the same homological class (this argument can be easily modified also for disconnected M).

In this paragraph we show that L can be framed if and only if $\bar{w}_2(L) = 0$. By the definition of $\bar{w}_2(L)$ this condition is necessary. In order to prove the sufficiency assume that $\bar{w}_2(L) = 0$. Since $n \geq 3$ and $\dim L = 2$, it follows that there is an orthonormal system of vector fields $f_1, \dots, f_{n-1}$ which are normal to L.

Since L^2 and M^{n+2} are orientable, it follows that the normal bundle to L is orientable. Fix an orientation of this bundle. Taking a unit vector field f_n orthogonal to $f_1, \dots, f_{n-1}$ and such that the basis $f_1, \dots, f_n$ is positive (with respect to the specified orientation of the bundle), we obtain the required framing.

Now the theorem follows from the equalities

$$\bar{w}_2(L) = w_2(M)|_L = w_2(M) \cdot [L] = w_2(M) \cdot \rho_2\alpha.$$

Here the first equality follows by the Wu formula of Stiefel–Whitney classes of the sum of two bundles: $w_2(M)|_L = w_2(L) + w_1(L) \cdot \bar{w}_1(L) + \bar{w}_2(L)$, in which $w_2(L) = w_1(L) = 0$ because L is an orientable 2-manifold (the first equality can also be proved directly). The second equality follows by the above geometric definition because L is connected (we identify $H_0(L;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \cong H_0(M;\mathbb{Z}/2\mathbb{Z})$). ∎

4. Proof of Theorem 1.1(a)

Take an element $\alpha \in H_1(M;\mathbb{Z})$. Let $L_1, L_2 \subset M$ be a pair of framed 1-submanifolds such that $\deg L_1 = \deg L_2$. Denote by $[L_1], [L_2] \in L_1(M)$ their classes. Since L_1 and L_2 are homologous, it follows by general position that there is an embedded 2-dimensional cobordism $L \subset M \times I$ (not framed) between them: $\partial L = L_1 \sqcup L_2$. Clearly, $[L_1] = [L_2]$ if and only if for some L the framing of ∂L extends to L. Since M is connected, it follows that we can consider only connected L.

Let us show that the framing of ∂L extends to that of L if and only if $\bar{w}_2(L) = 0$. By definition of the relative Stiefel–Whitney classes this condition is necessary. Let us prove the sufficiency. Assume that $\bar{w}_2(L) = 0$. Since $n \geq 3$ and $\dim L = 2$, it follows that the orthonormal system of the first $n-1$ vector fields of the framing of ∂L extends to L.

Since L and $M \times I$ are orientable, and L_1, L_2 are naturally orientable, it follows that there is an orientation of the normal bundle of L in $M \times I$ restricted to the given orientations on L_1 and L_2. So we can add one more unit vector field to the constructed orthonormal system on L to obtain a positive basis at each point of L (with respect to the specified orientation of L). So the required extension of the framing of ∂L to L has been constructed. ∎

Completion of the proof of Theorem 1.1(a) in the case when there exists $\beta \in H_2(M, \mathbb{Z}/2\mathbb{Z})$ such that $\beta \cdot w_2(M) \neq 0$. If $\bar{w}_2(L) = 0$ then there is nothing to prove. Assume now that $\bar{w}_2(L) = 1$. Here $\bar{w}_2(L) \in H_0(L; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, because L is connected. Further, we identify all groups $H_0(X; \mathbb{Z}/2\mathbb{Z})$ isomorphic to $\mathbb{Z}/2\mathbb{Z}$ with $\mathbb{Z}/2\mathbb{Z}$.

Let us construct a new cobordism L' between L_1 and L_2 such that $\bar{w}_2(L') = 0$. Let K be a connected orientable general position 2-submanifold realizing the class β. We may assume that $K \subset M \times \frac{1}{2}$. Then

$$\bar{w}_2(K) = |\Sigma \times I \cap K| = |\Sigma \cap K| = w_2(M) \cdot \beta = 1 \bmod 2.$$

Here $\Sigma \subset M \times \frac{1}{2}$ is a submanifold realizing the class $w_2(M)$; the first equality follows from geometric definition above. Put $L' = L \sharp K$ ($L \cap K = \emptyset$ by general position). By Claim 4.1 below $\bar{w}_2(L') = \bar{w}_2(L) + \bar{w}_2(K) = 0$, and this case of the theorem is proved. ∎

Claim 4.1: *Suppose that $K^2, L^2 \subset M^{n+2}$ is a pair of disjoint connected orientable submanifolds and a frame of K and L is given on ∂K and ∂L, respectively. Then $\bar{w}_2(K \sharp L) = \bar{w}_2(K) + \bar{w}_2(L)$, where the groups $H_0(X; \mathbb{Z}/2\mathbb{Z})$ are identified with $\mathbb{Z}/2\mathbb{Z}$ for $X = K \sharp L$, K and L.*

Proof of Claim 4.1: Take a pair of small 2-disks $k \subset K$ and $l \subset L$. Let $kl \cong S^1 \times I$ be a narrow tube such that $\partial kl = \partial k \sqcup \partial l$ and kl is tangent to both disks k and l. Fix a trivial frame of k and l (and, consequently, of ∂k and ∂l).

By the above geometric definition it follows easily that $\bar{w}_2(K \sharp L) = \bar{w}_2(K - k) + \bar{w}_2(kl) + \bar{w}_2(L - l)$. On the other hand, one can check analogously that $\bar{w}_2(K) = \bar{w}_2(K - k) + \bar{w}_2(k)$ and $\bar{w}_2(L) = \bar{w}_2(L - l) + \bar{w}_2(l)$. Since $\bar{w}_2(kl) = \bar{w}_2(k) = \bar{w}_2(l) = 0$, it follows that $\bar{w}_2(K \sharp L) = \bar{w}_2(K) + \bar{w}_2(L)$. ∎

COMPLETION OF THE PROOF OF THEOREM 1.1(a) IN THE CASE WHEN FOR EACH $\beta \in H_2(M,\mathbb{Z}/2\mathbb{Z})$ WE HAVE $\beta \cdot w_2(M) = 0$. It suffices to show that for fixed $[L_1]$ the map $[L_2] \mapsto w_2(L)$ is well-defined and is a bijection $\deg^{-1}\alpha \to \mathbb{Z}/2\mathbb{Z}$.

Let us prove that the map is well-defined. Let L_1' and L_2' be a pair of framed submanifolds of M framed cobordant to L_1 and L_2, respectively. Let L' be a (not framed) cobordism between them. It suffices to prove that $w_2(L) = w_2(L')$ in case when L_1 and L_1', L_2 and L_2', L and L' are in general position.

Assume that $L_1, L_1' \subset M \times 1$, $L_2, L_2' \subset M \times 0$ and $L, L' \subset M \times [0,1]$. Change the sign of the first vector field belonging to the framings of L_1' and L_2'. Denote the obtained framed submanifolds by $-L_1'$ and $-L_2'$, respectively.

Denote by $\bar{w}_2(-L')$ the relative Stiefel–Whitney class of L with the reversed framing of $\partial L'$. Then $\bar{w}_2(-L') = -\bar{w}_2(L')$. Further, both $L_1 \sqcup (-L_1')$ and $L_2 \sqcup (-L_2')$ are framed cobordant to zero, i.e., to an empty submanifold. Let $L_+ \subset M \times [1,+\infty)$ and $L_- \subset M \times (-\infty, 0]$ be the corresponding framed cobordisms. Then $\bar{w}_2(L_+) = \bar{w}_2(L_-) = 0$.

By general position $L \cap L' = \emptyset$. Denote $K = L \cup L_+ \cup L' \cup L_-$. By the above geometric definition it follows easily that

$$\bar{w}_2(K) = \bar{w}_2(L) + \bar{w}_2(L_+) + \bar{w}_2(-L') + \bar{w}_2(L_-) = \bar{w}_2(L) - \bar{w}_2(L').$$

It suffices to show that $\bar{w}_2(K) = 0$. Let β be the cohomological class of image of K under the projection $M \times \mathbb{R} \to M$. Analogously to the proof of the previous case of the theorem we see that $\bar{w}_2(K) = \bar{w}_2(M) \cdot \beta = 0$, hence $w_2(L) = w_2(L')$ and our map $\deg^{-1}\alpha \to \mathbb{Z}/2\mathbb{Z}$ is well-defined.

Now let us prove that our map is injective. It suffices to show that if L_2' is a framed 1-submanifold and L' is a connected 2-dimensional embedded cobordism (not framed) between L_1 and L_2' such that $\bar{w}_2(L) = \bar{w}_2(L')$, then $[L_2] = [L_2']$.

Indeed, we may assume that $L_1 \subset M \times 0$, $L_2 \subset M \times 1$, $L_2' \subset M \times (-1)$, $L \subset M \times [0,1]$ and $L' \subset M \times [-1,0]$. Then $L \cup L'$ is a cobordism between L_2 and L_2'. By the above geometric definition it follows that $\bar{w}_2(L \cup L') = \bar{w}_2(L) + \bar{w}_2(L') = 0$, hence $L \cup L'$ can be framed. So our map $\deg^{-1}\alpha \to \mathbb{Z}/2\mathbb{Z}$ is injective.

Let us prove that our map is surjective. It suffices to show that some $[L_2]$ is mapped to 1. Since M is orientable, it follows there exists a framing f_1 of L_1. Fix a homeomorphism $L_1 \cong S^1$.

Denote by $f_1(x)$ the choice of the framing at the point $x \in S^1$. Take a map $\varphi\colon S^1 \to SO(n)$ realizing a nonzero element of $\pi_1(SO(n)) \cong \mathbb{Z}/2\mathbb{Z}$ (which is true because $n \geq 3$). Define a new framing f_2 of L_1 by the formula $f_2(x) = \varphi(x) f_1(x)$.

The obtained framed submanifold is the required submanifold L_2. Indeed, take $L = L_1 \times I$. Then $\bar{w}_2(L) = 1$. Indeed, assume the converse. Then the frames of L_1 and L_2 can be extended to the frame of $L_1 \times I$. This frame gives the homotopy between φ and the constant map in $SO(n)$, which contradicts the choice of φ. This contradiction completes the proof of Theorem 1.1(a). ∎

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